(3) $f(z) = \frac{z+1}{z-1}$ has a singularity at $z=1$. We can find a Taylor series on the disk $|z|<1$ and a Laurent series on the annulus $1<|z|<\infty$.

On $|z|<1$:
$$\frac{z+1}{z-1} = -(z+1)\frac{1}{1-z} = -(z+1)\sum_{n=0}^{\infty} z^n$$
$$= -\sum_{n=0}^{\infty} z^{n+1} - \sum_{n=0}^{\infty} z^n$$
$$= -1 - 2\sum_{n=0}^{\infty} z^{n+1}$$
$$= -1 - 2\sum_{n=1}^{\infty} z^n.$$

On $1<|z|<\infty$: this condition implies $|\frac{1}{z}|<1$. We have
$$\frac{z+1}{z-1} = \frac{z}{z}\cdot\frac{1+\frac{1}{z}}{1-\frac{1}{z}} = \frac{1}{1-\frac{1}{z}} + \frac{1}{z}\left(\frac{1}{1-1/z}\right)$$
$$= \sum_{n=0}^{\infty}\frac{1}{z^n} + \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}$$
$$= 1 + 2\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$
$$= 1 + 2\sum_{n=1}^{\infty}\frac{1}{z^n}.$$

(4) $f(z) = \frac{1}{(z-z_0)^{n+1}}$, $n \geq 0$. This is analytic on the annulus $0<|z-z_0|<\infty$. In fact, $f(z)$ is already a Laurent series. We will compute
$$\frac{1}{2\pi i}\int_C \frac{1}{(z-z_0)^{(n+1)-m}}\,dz$$

for any $m \geq 0$ where C is any simple closed contour about z_0. By Laurents theorem

$$b_{m+1} = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{-(m+1)+1}} \, dz$$

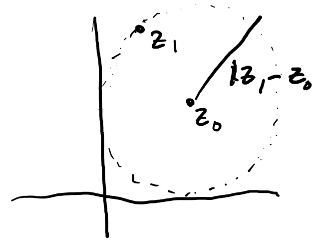
$$= \frac{1}{2\pi i} \int \frac{1}{(z-z_0)^{(n+1)-m}} \, dz$$

But $b_{m+1} = \begin{cases} 1, & m=n \\ 0, & \text{otherwise.} \end{cases}$

//

Absolute & Uniform Convergence

Theorem (Power Series Converge Absolutely) If a power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges when $z = z_1$, then it converges absolutely on the disk $D_R(z_0)$, $R = |z_1 - z_0|$.



Proof. Assume the series converges at z_1. Then the sequence $a_n(z-z_0)^n$ is bounded. Choose $M>0$ so that $|a_n(z-z_0)^n| \leq M$ for all $n \in \mathbb{N}$. Now let $z \in D_R(z_0)$ so that $|z-z_0| < R = |z_1 - z_0|$. Write

$$\rho = \frac{|z-z_0|}{|z_1-z_0|}.$$

Then $$|a_n(z-z_0)^n| = |a_n(z_1-z_0)^n| \left| \frac{(z-z_0)}{(z_1-z_0)} \right|^n$$

$$\leq M \rho^n.$$

But the series $\sum_{n=0}^{\infty} M\rho^n$ is a convergent geometric series since $\rho < 1$. Hence, by the comparison test, the series $\sum_{n=0}^{\infty} |a_n (z - z_0)^n|$ converges. ∎

The theorem asserts that if a series converges at a point $z_1 \neq z_0$, then it converges on a disk $D_{|z_1 - z_0|}(z_0)$. The largest disk with this property is called the disk of convergence or circle of convergence. According to the theorem, a series does not converge at any point outside its disk of convergence.

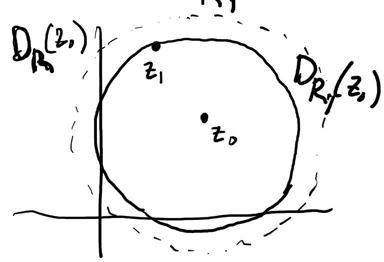
$\boxed{\text{Definition (Uniform Convergence of Series)}}$ Consider a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ with disk of convergence $D_R(z_0)$.

Let S be a region in the disk. We say that the series converges uniformly on S if for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \text{ and } z \in S \text{ implies } |\rho_n(z)| < \varepsilon.$$

In other words, $n_0 = n_0(\varepsilon)$ depends only on ε and not on z.

$\boxed{\text{Theorem (Uniform Convergence of Series)}}$ If z_1 is a point inside the disk of convergence $D_R(z_0)$ of a series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, then the series converges uniformly on the closed disk $\overline{D_{R_1}(z_0)}$ where $|z_1 - z_0| = R_1$.



Proof. By the preceding theorem, the series

$$\sum_{n=0}^{\infty} |a_n (z_1 - z_0)^n|$$

converges. Write the remainders of each series:

$$\rho_N(z) = \lim_{m \to \infty} \sum_{n=N}^{m} a_n (z - z_0)^n$$

$$\sigma_N = \lim_{m \to \infty} \sum_{n=N}^{m} |a_n (z_1 - z_0)^n|,$$

Consider $z \in \overline{D_{|z_1 - z_0|}(z_0)}$. Then $|z - z_0| \leq |z_1 - z_0|$. Hence,

$$|\rho_N(z)| = \lim_{m \to \infty} \left| \sum_{n=N}^{m} a_n (z - z_0)^n \right|$$

$$\leq \lim_{m \to \infty} \sum_{n=N}^{m} |a_n| \, |z - z_0|^n$$

$$\leq \lim_{m \to \infty} \sum_{n=N}^{m} |a_n| \, |z_1 - z_0|^n = \sigma_N.$$

Let $\varepsilon > 0$. Choose $n_0(\varepsilon) \in \mathbb{N}$ such that $N \geq n_0(\varepsilon)$ implies $|\sigma_N| < \varepsilon$. Hence, $N \geq n_0(\varepsilon)$ and $z \in \overline{D_{|z_1 - z_0|}(z_0)}$ implies $|\rho_N(z)| \leq \sigma_N < \varepsilon$.

$\blacksquare$

$\boxed{\text{Theorem (Continuity of Power Series)}}$ A power series

$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is a continuous function on its disk of convergence.

Proof. Let $D_R(z_0)$ be the disk of convergence and let $z_1 \in D_R(z_0)$. Let $\varepsilon > 0$. Since the power series converges uniformly, choose $N(\varepsilon) \in \mathbb{N}$ such that for all $z \in \overline{D_{|z_1 - z_0|}(z_0)}$,

$$N \geq N(\varepsilon) \implies |\rho_N(z)| < \frac{\varepsilon}{3}.$$

Also, since $S_N(z)$ is a polynomial for each $N \in \mathbb{N}$, it is a continuous function. Fix $N_0 = N(\varepsilon)+1$. Choose $\delta > 0$ such that

$$|z - z_1| < \delta \implies |S_{N_0}(z) - S_{N_0}(z_1)| < \frac{\varepsilon}{3}.$$

Then $|z - z_1| < \delta$ implies

$$\begin{aligned} |S(z) - S(z_1)| &= |S_{N_0}(z) + \rho_{N_0}(z) - (S_{N_0}(z_1) + \rho_{N_0}(z_1))| \\ &\leq |S_{N_0}(z) - S_{N_0}(z_1)| + |\rho_{N_0}(z)| + |\rho_{N_0}(z_1)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

▨

P2(b), P6

Theorem (Integrating Power Series) Let C be any contour interior to the disk of convergence of the power series

$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Let $g(z)$ be any function continuous on C. Then

$$\int_C g(z) S(z)\, dz = \sum_{n=0}^{\infty} a_n \int_C g(z) (z - z_0)^n dz.$$

Proof. Denote by $\sigma_N(z)$ the remainder of the series on the right. Write $\rho_N(z) = S(z) - S_N(z)$. We have

$$\begin{aligned} \sigma_N(z) &= \int_C g(z) S(z) dz - \sum_{n=0}^{N-1} a_n \int_C g(z) (z - z_0)^n dz \\ &= \int_C g(z) \left(S(z) - \sum_{n=0}^{N-1} a_n (z - z_0)^n \right) dz \end{aligned}$$

$$= \int_C g(z)\, \rho_N(z)\, dz.$$

Let $\varepsilon > 0$. Since g is continuous on C, choose $M > 0$ such that

$$|g(z)| \leq M \quad \text{for all } z \in C.$$

Since $S(z)$ is uniformly convergent on its disk of convergence $D_R(z_0)$, choose $N(\varepsilon) \in \mathbb{N}$ such that for all $z \in D_R(z_0)$,

$$N \geq N(\varepsilon) \implies |\rho_N(z)| < \frac{\varepsilon}{M \cdot \text{length}(C)}.$$

Then by the Triangle Ineq. for contour integrals,

$$|\sigma_N(z)| = \left| \int_C g(z)\, \rho_N(z)\, dz \right|$$

$$\leq \max_{z \in C} |g(z)| \cdot |\rho_N(z)| \cdot \text{length}(C)$$

$$\leq M \cdot \frac{\varepsilon}{M\, \text{length}} \cdot \text{length}(C) = \varepsilon.$$

This proves that $\lim_{N \to \infty} \sigma_N(z) = 0$.

$\blacksquare$

$\boxed{\text{Corollary (Power Series are Analytic)}}$ A power series

$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is an analytic function on its disk of convergence.

Proof. By one of the theorems, $S(z)$ is continuous on its disk of convergence. Let C be any closed contour lying inside the disk of convergence. Then

$$\int_C S(z)\,dz = \sum_{n=0}^{\infty} a_n \int_C (z-z_0)^n\,dz \qquad \text{(by the theorem)}$$

$$= \sum_{n=0}^{\infty} a_n \cdot 0 \qquad \left(\text{since } (z-z_0)^n \text{ has an anti-derivative}\right)$$

$$= 0.$$

By Morera's theorem, $S(z)$ is an analytic function! $\blacksquare$

P4

$\boxed{\text{Example}}$ The function

$$f(z) = \begin{cases} \dfrac{\sin z}{z} & , z \neq 0 \\ 1 & , z = 0 \end{cases}$$

is entire. For any $z \in \mathbb{C}$, we can write

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}.$$

When $z \neq 0$, we have

$$\frac{\sin z}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!}.$$

But when $z = 0$,

$$1 = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!}$$

Hence, $f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!}$ for any $z \in \mathbb{C}$. //

$\boxed{\text{Theorem (Differentiating Power Series)}}$ A power series

$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

can be differentiated term-by-term. Precisely, at each point interior to the disk of convergence,

$$S'(z) = \sum_{n=1}^{\infty} n \cdot a_n (z - z_0)^{n-1}.$$

Proof. Let $D_R(z_0)$ be the disk of convergence and let $z \in D_R(z_0)$. Let C be a simple closed positively oriented contour interior to $D_R(z_0)$ and surrounding z. Then

$$S'(z) = \frac{1}{2\pi i} \int_C \frac{S(w)}{(w-z)^2}\, dw \quad \left(\text{Cauchy's Int. formula}\right)$$

$$= \int_C g(w)\, S(w)\, dw \quad \left(g(w) = \frac{1}{2\pi i} \cdot \frac{1}{(w-z)^2} \right)$$

$$= \sum_{n=0}^{\infty} a_n \int_C g(w) (w - z_0)^n \, dz \quad \left(\text{Integrating Power series}\right)$$

$$= \sum_{n=0}^{\infty} a_n \cdot \frac{1}{2\pi i} \int_C \frac{(w - z_0)^n}{(w-z)^2}\, dz$$

$$= \sum_{n=0}^{\infty} a_n \left. \frac{d}{dw} (w - z_0)^n \right|_{w=z} \quad \left(\text{Cauchy Integral Formula}\right)$$

$$= \sum_{n=0}^{\infty} a_n\, n\, (z - z_0)^{n-1}.$$

$\blacksquare$